Quantum frustrated magnets

Heisenberg models,
spin liquids, topological order, anyons,
fractionalization, entanglement, ...

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Outline

- Classical & quantum frustration
- 'Conventional' phases: Néel, VBC, …
- Quantum spin liquids
  - Fractionalization (anyon) & topological order
  - Example: Z_2 liquid
  - Entanglement
  - Gapless liquids

Some examples of

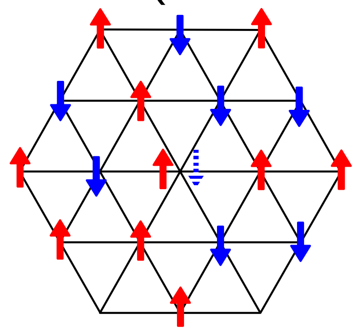
CLASSICAL FRUSTRATION

Classical frustration (1)

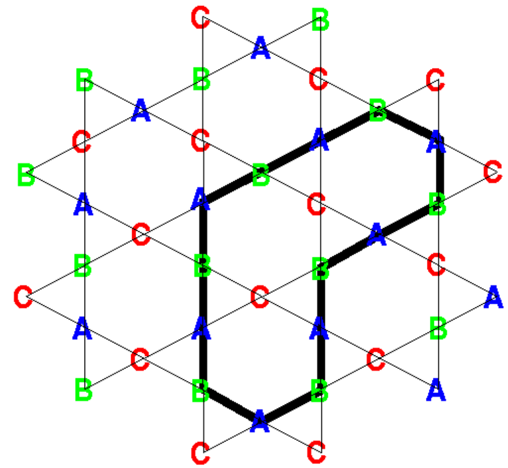
Frustration → **many low-energy states** (sometimes degenerate)

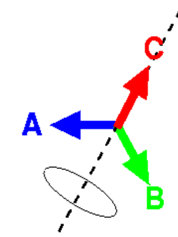
Examples:

- Ising AF triangular lattice
  Wannier 1950



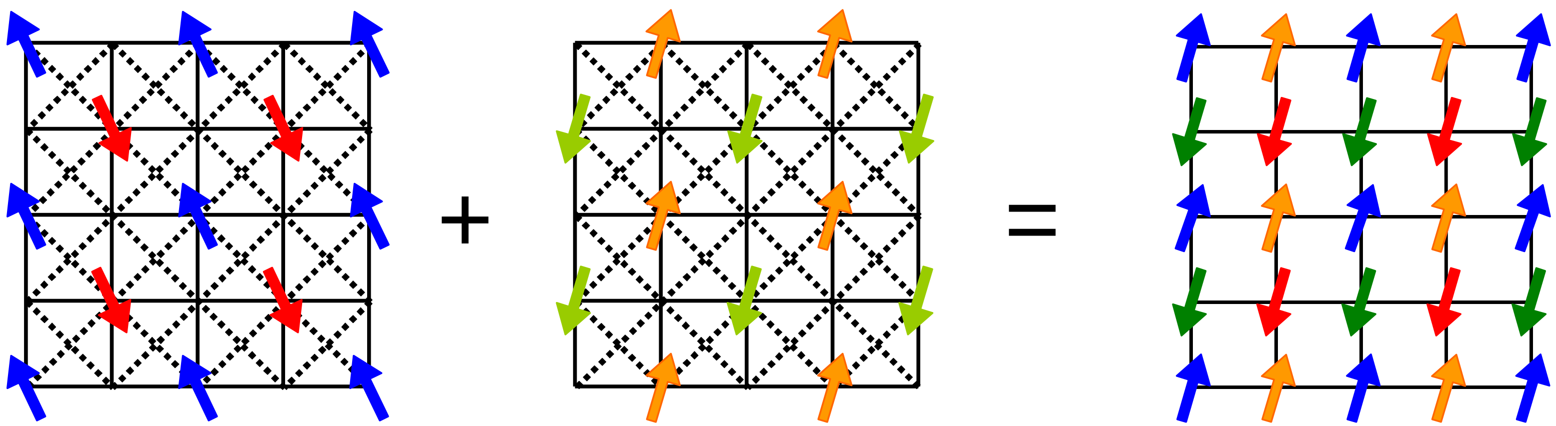
- O(3) Kagome antiferromagnet





- $J_1 - J_2$ O(3) model

Nb: even larger deg. at $J_2 = 0.5 J_1$



$0 < ½J_1 < J_2$

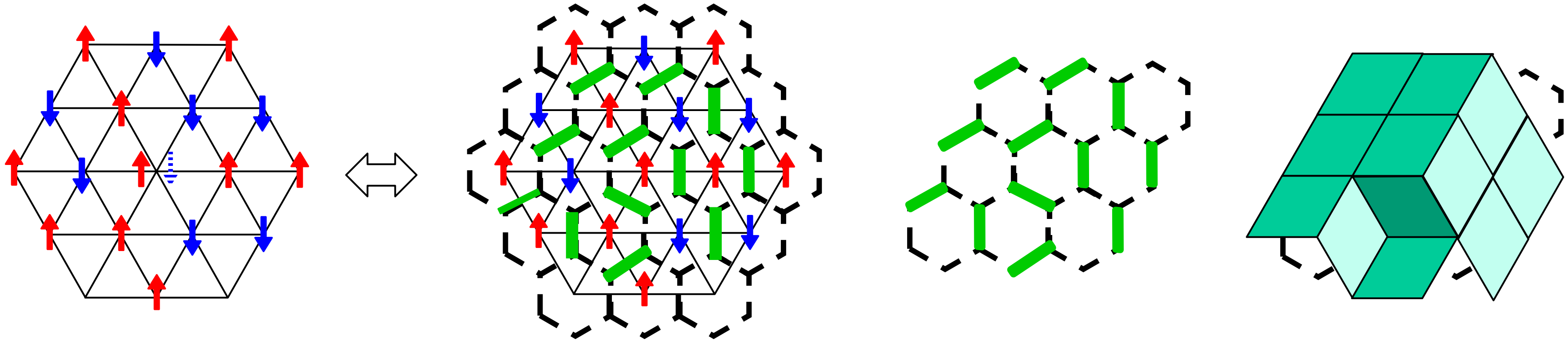
$\theta = \pi$

Classical frustration (2)

Frustration → **new/emerging low energy degrees** of **freedom**

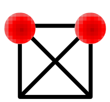
Examples:

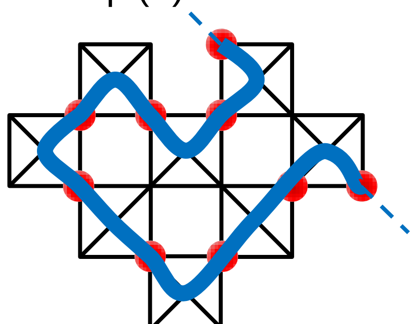
- Ising AF triangular → dimers on the honeycomb lattice → 2D surface/height model



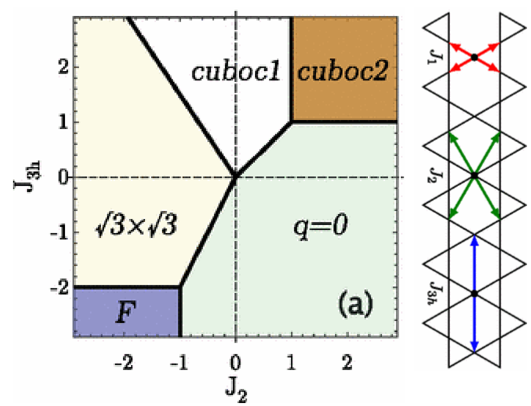
- Ising checkboard

$H = \left(\sum_{\boxtimes} \sigma_i^z\right)^2$: Minimization → 2 up (●) & 2 down on each crossed-square

 → loops 



- Local constraint ↔ gauge theory ($div\,\vec{E} = 0$)
- Frustration → **competing orders**
  ex: classical phase diagram of the O(3) $J_1 - J_2 - J_3$ model on kagome



Messio, Bernu & Lhuillier, PRL 2012

How to make a frustrated system

QUANTUM ?

Quantum dynamics

Some ways to add quantum dynamics to classical/frustrated spin system

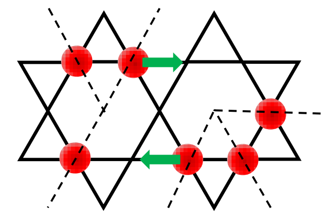
- **Transverse field** $H = \sum_{ij} J_{ij}^z \sigma_i^z \sigma_j^z + h \sum_i \sigma_i^x$

  ex.: Frustrated Ising + transverse field: Moessner & Sondhi PRL 2001

- **"xy" Exchange** $H = \sum_{ij} J_{ij}^z \sigma_i^z \sigma_j^z + J^{\perp} \sum_{<ij>} \left(S_i^+ S_j^- + S_i^- S_j^+ \right)$

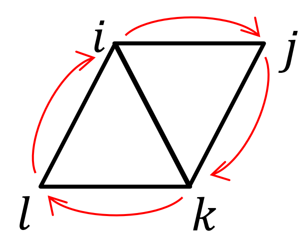
  ex.: of quantum spin liquid in an easy-axis XXZ model:
  Balents, Fisher & Girvin, PRB 2002

$$\mathcal{H}_0 = J_z \sum_{\hexagon} (S^z_{\hexagon})^2$$
$$\mathcal{H}_1 = J_\perp \sum_{\hexagon} [(S^x_{\hexagon})^2 + (S^y_{\hexagon})^2 - 3]$$



- Cyclic **ring exchange**

$$H = \cdots + K \sum_{<ijkl>} P_{ijkl} + h.c.$$



  Hubbard model: $K \sim \frac{t^4}{U^3}$

- …

Methods ?

- Strongly interacting many-body problems, no obvious “small” parameter, no general method which works in all/most cases.
- Numerics (→ A. LAUCHLI’s):
  - Exact Diagonalizations
  - Variational MC (→ F. BECCA’s),
  - DMRG ($d = 1$ & $d = 2$)
  - Tensor-network based approaches (→ N. SCHUCH’s)
  - Series expansion
  - coupled clusters expansions
  - QMC (→ F. Assaad). Sign problem → specific models only (Sandvik’s $J - Q$, PRL 2007, etc.)
- Analytics:
  - Large-S
    Large-N (Abrikosov Fermions, Schwinger bosons, …)
  - Effective models, toy models & ansatze wave-functions: dimers, string-nets, RK points, …
  - field theory & renormalization group, gauge theory mappings, …

What are the possible

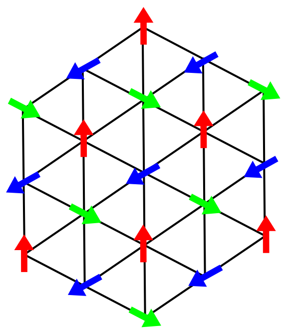
PHASES AT T=0 ?

Results from 20th century

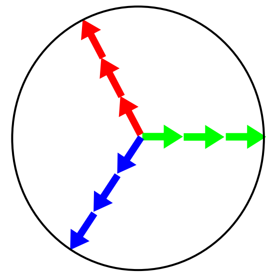
CONVENTIONAL PHASES

Magnetic LRO & broken continuous symmetry

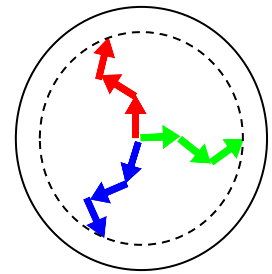
- A canonical example, the triangular Heisenberg antiferromagnet



Classical

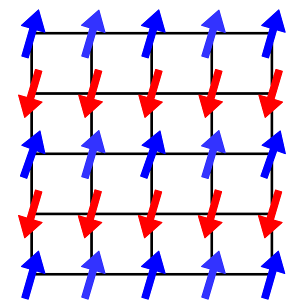


Quantum



Jolicœur & Le Guillou, 1989
Huse & Elser 1988
Bernu, Lhuillier & Pierre, 1992

- Gapless (spin waves=Nambu-Goldstone)

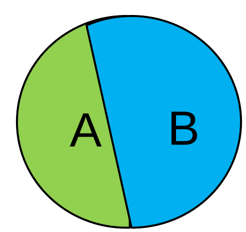


- Possibility of « order-by-disorder »

  ex.: $J_1 - J_2$ on square lattice
  Collinear mag. order for $J_2 > 0.5J_1$, although the classical g.-state is degenerate.
  Chandra, Coleman & Larkin, PRL 1990

- More complex continuous sym. breaking are possible: nematic orders
- Long-wavelength fluctuations → entanglement

  Additive $\log(L)$ corrections to the area law in the entanglement entropy
  $S(\mathrm{L}) \sim \alpha L + \frac{N_G(d-1)}{2}\log(L)$, where N_G is the number of Goldstone modes

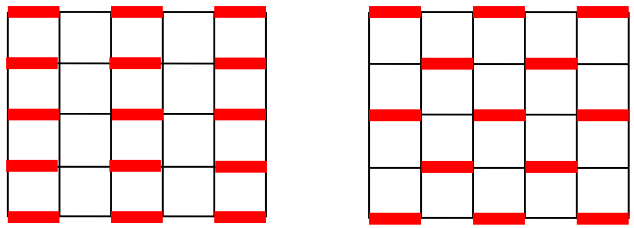


$|\psi\rangle$

Metlitski & Grover, arXiv:1112.5166; Kallin, Hastings, Melko & Singh, PRB 2011
Luitz, Plat, Alet, & Laflorencie, PRB 2015; Laflorencie, Luitz, & Alet, PRB 2015

Valence-bond crystals

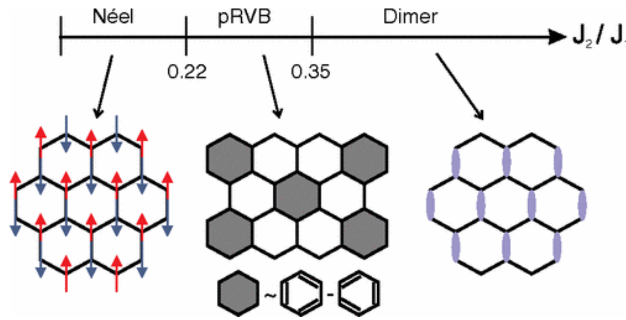
- Spatial spontaneous symmetry breaking. Short-ranged spin-spin correlations
  - Example 1



Heisenberg model & 4-spin "ring" exchange
Läuchli, Domenge, Lhuillier, Sindzingre & Troyer, PRL 2005

  - Example 2

$J_1 - J_2$ spin-$\frac{1}{2}$ Heisenberg antiferromagnet on a honeycomb



From: Ganesh, Van den Brink & Nishimoto, PRB 2013
See also: Z. Zhu, Huse, & White, PRB 2013

- Magnetic excitations are gapped $S = 1$ magnons (spin-$\frac{1}{2}$ spinons are confined)

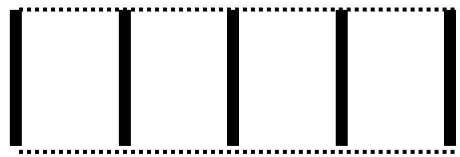
Question: what happens if you 1) break a singlet into a triplet (=two nearby spin-1/2)
2) try to separate these spin-1/2 ?

- Possibility of interesting continuous phase transition between Néel & VBC: "deconfined" criticality
(requires to go beyond standard Landau-Ginzburg theory of phase transitions)
T. Senthil *et al*., Science **303**, 1490 (2004)

- Can we have some states *without any broken symmetry* ?

Quantum paramagnets

- Adiabatically connected to some “decoupled” limit (or band insulator)

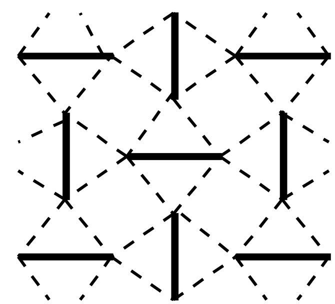


Strong explicit dimerization
J(—) >> J’(⋯)

- Gap & short-range (connected) correlations, no spon. broken sym. unique ground-state.

- “weakly entangled” states (exists a product-state limit)
- A famous example: $SrCu_2(BO_3)_2$
  (very interesting magnetization curve with many plateaus)
- Other examples:
  - CaV_4O_9
  - $TlCuCl_3$
  - $Rb_2Cu_3SnF_{12}$, …

J ▬ ≈ 100 K
J’- - - - ≈ 68 K

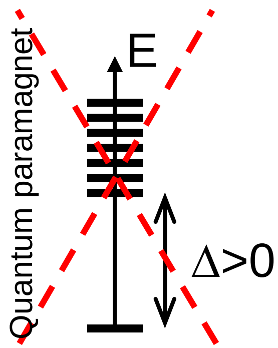


Kageyama *et al. (1999)*

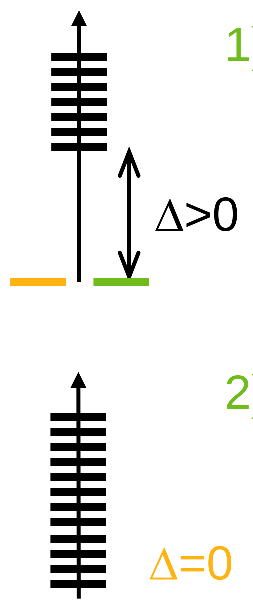
Lieb-Shultz-Mattis-Hastings theorem

A featureless quantum paramagnet is impossible if the system has a half odd-integer spin per unit cell (=Mott insulator).

Lieb-Schultz-Mattis Theorem for spin chains ($d = 1$): 1961
Recent proof valid in any dimension ($d > 1$) : Hastings, PRB 2004
See also: Affleck 1988; Bonesteel 1989; Oshikawa PRL 2000; Nachtergaele & Sims 2007



Conditions:
- half-odd-integer spin in the unit cell
- short-range interactions
- Global U(1) symmetry: $[S^z_{tot},H]=0$
- dimensions $L_1 \times L_2 \times \ldots \times L_D$ with $L_2 \times \ldots \times L_d$=odd
- periodic bound. conditions in direction 1
- thermodynamic limit



1) Ground-state degeneracy
   - a- Conventional broken symmetry (ex.:VBC)
   - b- “Topological” degeneracy

“**spin liquids**”

2) Gapless spectrum
   - a- Continuous broken sym. (ex.: Néel order)
   - b- Critical phase (or crit. point)

QUANTUM SPIN LIQUIDS

What is a quantum spin liquid ?

Several possible definitions:

- def 0: Spin system which remains "**disordered**" (=no SSB) down to $T = 0$
  - = the classical point of view
  - It would include quantum paramagnets
    … unless one demands a half-odd-integer spin/cell
  - it would exclude systems where some conventional order could coexist with some QSL/topological order physics… (like some chiral spin liquids)

- def 1: A phase which admits **no "product-state**" limit

  Recent review developing this "entanglement" point of view:
  Savary & Balents arXiv:1601.03742 (see also: Balents, Nature 2010)

- def 2: A phase which sustains some **fractionalized** excitations

  Restricted to gapped QSL

Gapped QSL in 2D:

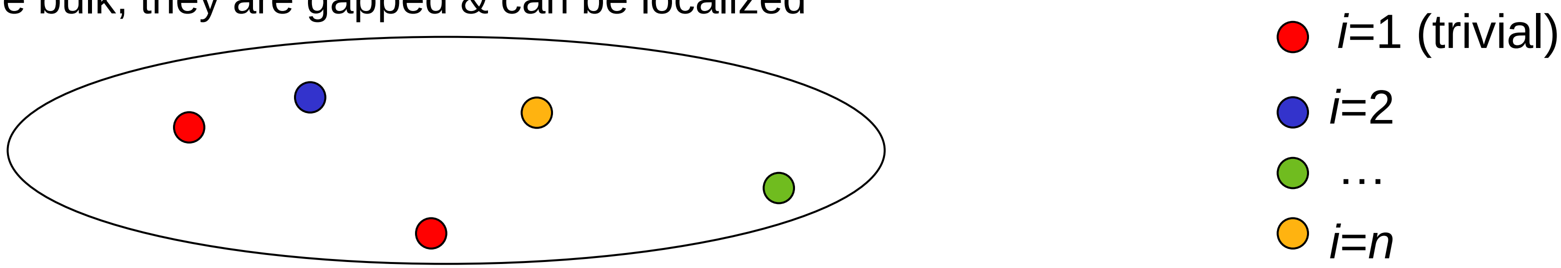
FRACTIONALIZATION & TOPOLOGICAL ORDER

Anyons in 2d

Phenomenology for the elementary excitations in a “topological” phase

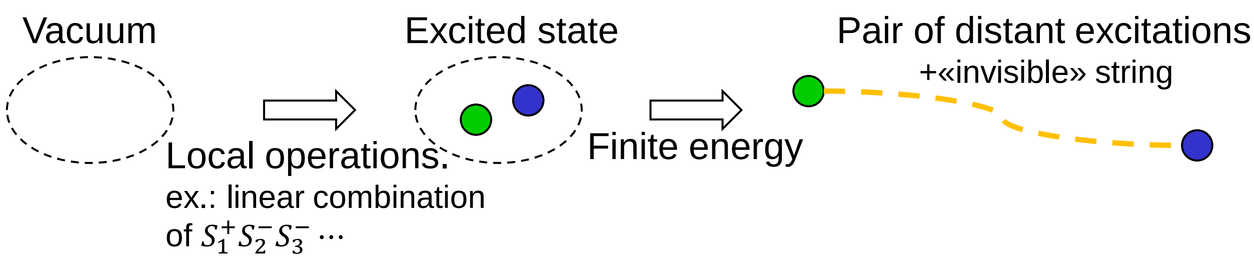
- n (topologically distinct) types of elementary excitations.

They live in the bulk, they are gapped & can be localized

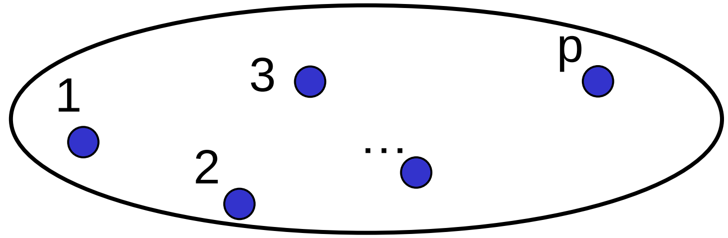


- By acting locally, one can only create/destroy **pairs** of particles

(except the topologically “trivial”-type particle... Allowed pairs are given in terms of the so-called fusion rules)



Anyons quantum dimensions



- p localized quasiparticles → degeneracy:
  - #of states ~ $(d_i)^p$ when $p \gg 1$
  - These states **are locally indistinguishable**
  - d_i : so-called **quantum dimension** (not necessarily an integer) of the quasiparticles of type i

- For Abelian anyons, $d_i = 1$ (no degeneracy)

- non-Abelian $d_i > 1$
  - example 1: Ising anyon " σ "
    (chiral p-wave superconductors, FQHE @ $\nu = \frac{5}{2}$ & Moore-Read Paffian state)

$$\dim_{2N} = 2^N \rightarrow d_\sigma = \sqrt{2}$$

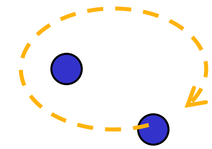
  - example 2: Fibonacci anyon « τ »
    (FQHE & Read-Rezayi state)

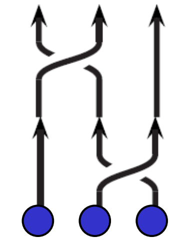
$$\dim = 1,2,3,5,8,\cdots = \text{Fibonacci suite} \rightarrow d_\tau = \frac{1+\sqrt{5}}{2}$$

  - Short introduction to anyons, $SU(2)_k$, etc: Trebst, Troyer, Wang & Ludwig, 2008

Anyon statistics

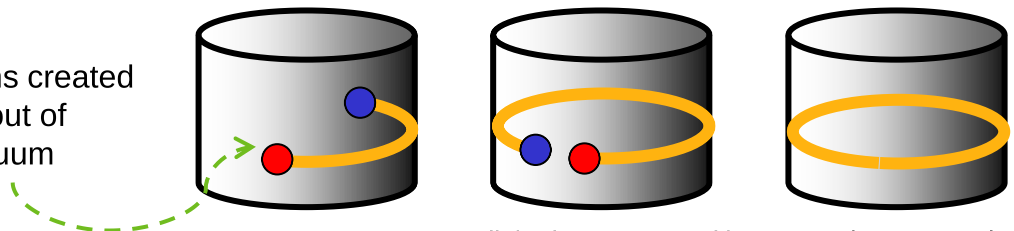
- Braiding :
  - possible non-trivial **phase** (if $d_i = 1$)
  - or **matrix** if some $d_i > 1$
  - Topological quantum computation
    Nayak *et al.*, Rev Mod Phys 2008





- Topological degeneracy of the ground-state

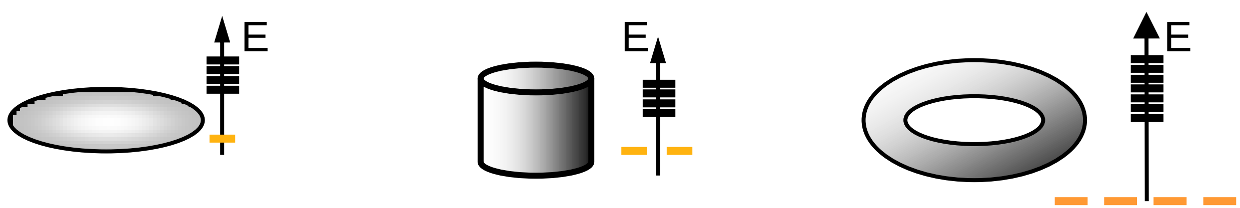
2 anyons created locally out of the vacuum



Topological degeneracy
X.-G. Wen 1991
See also Oshikawa & Senthi PRL 2006

adiabatic process → New ground-state, topologically distinct from the initial vacuum

ex. : $\mathbb{Z}_2$ liquid



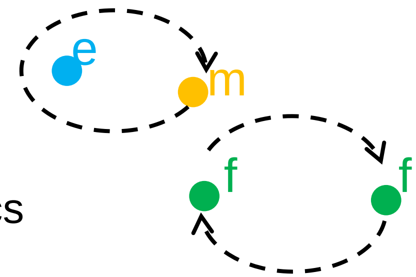
Generally: degeneracy on a torus = number of quasiparticle types

Simplest example: $\mathbb{Z}_2$ liquid

- 4 topological types of excitations (all with quantum dimension $d_i = 1$)
  - **1** : Trivial particle
  - **e** : “electric charge”
  - **m**: “magnetic charge”
  - **f** : **e-m** pair

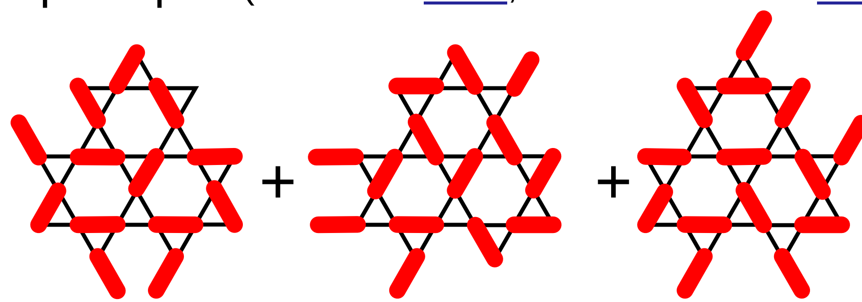
$e^{i\pi}$ Mutual statistics

Fermionic self-statistics



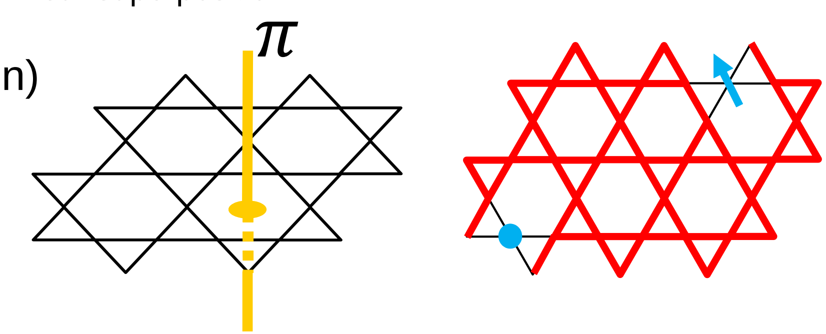
- Realization in a short-range RVB spin liquid (Anderson 1973; Read & Sachdev 1991; …)

 $= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

$|\psi_{\rm RVB}\rangle \sim$  $+ \ldots$

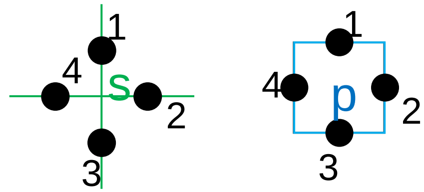
The lattice symmetries (translation etc.) are restored in this linear superposition

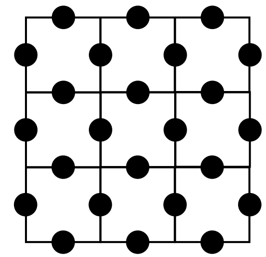
  - **1** : Any local excitation (example: a $S = 1$ magnon)
  - **e** : spin-$\frac{1}{2}$ spinon (or hole)
  - **m**: vison (π-flux vortex in the singlet amplitudes)
  - **f** : spinon-vison bound-state



Examples of $\mathbb{Z}_2$ liquids

- Kitaev's Toric code, arXiv 1997 (=Ann. of Phys. 2003)



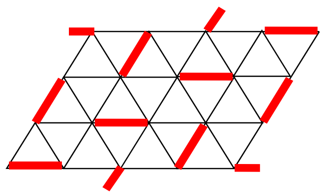


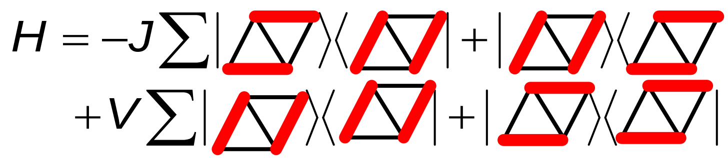
$$H = -\sum_s A_s - \sum_p B_p$$

$$A_s = \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x$$

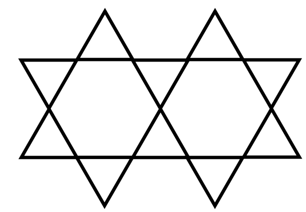
$$B_p = \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z$$

- Quantum dimer models
  - triangular lattice: Moessner & Sondhi PRL 2001,





  - kagome: GM, Serban & Pasquier PRL 2002 (exactly solvable & similar to the toric code)



- Balents Fisher Girvin, PRB 2002

  mapping onto a QDM@triangular with 3 dimers touching each site
  See also : Isakov *et al.* 2006; Isakov *et al.* 2011

$$\mathcal{H}_0 = J_z \sum_{\hexagon} (S^z_{\hexagon})^2$$

$$\mathcal{H}_1 = J_\perp \sum_{\hexagon} [(S^x_{\hexagon})^2 + (S^y_{\hexagon})^2 - 3]$$

- Several spin-½ antiferromagnetic **Heisenberg models** are $\mathbb{Z}_2$-liquid candidates. For instance:

DMRG studies

  - **J_1-only @Kagome**:
    S. Yan, D. Huse & S. White, Science 2011;
    S. Depenbrock, I. P. McCulloch, &U. Schollwöck PRL 2012;
    H.-C. Jiang, Z. Wang & L. Balents, Nature 2012
  - …
  - **$J_1 - J_2$@triangular**
    Z. Zhu & S. R. White, PRB 2015
    W.-J. Hu, S.-S. Gong, W. Zhu & D. N. Sheng, PRB 2015

Symmetry-enriched topological phases

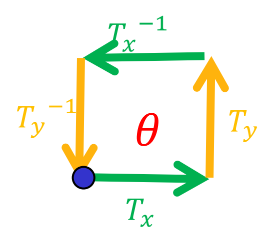
- Topological order (fusion rules, quantum dimension, topological degeneracy, …) is robust against **all local perturbations**. Symmetries are not required.

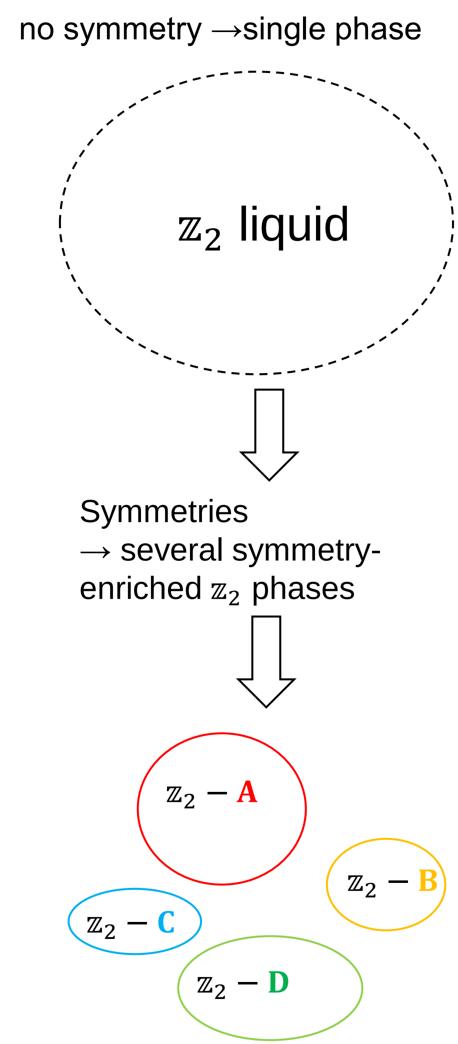
- In presence of symmetries (lattice translations, spin-rotations, …) a given topological phase may give rise to several **symmetry-enriched topological (SET)** phases.

- Physical states (containing necessarily an even number of anyons) transform according to linear representations of the symmetry group.
But, when defined on *single* anyon (subtle!), the symmetry operations correspond to a **projective representation** $\boldsymbol{U}$:

$$U_g U_h = e^{i\theta(g,h)} U_{gh}$$

- The different SET phases correspond to different projective representations ("symmetry fractionalization").
Example with 2 translations:
$T_x T_y {T_x}^{-1} {T_y}^{-1} = 1$ but $U_{T_x} U_{T_y} U_{{T_x}^{-1}} U_{{T_y}^{-1}} = e^{i\theta}$
θ: topological invariant of the phase





- A few references on $\mathbb{Z}_2$ liquids & SET:

X.-G. Wen, PRB 2002 (Projective symmetry group & parton construction); F. Wang & A. Vishwanath, PRB 2006 (Schwinger Bosons on triangular & kagome lattices); M. Essin & M. Hermele, PRB 2013; C.Y. Huang, X. Chen & F. Pollmann, PRB 2014; M. Zaletel, Y.-M. Lu & A Vishwanath, arXiv:1501.01395

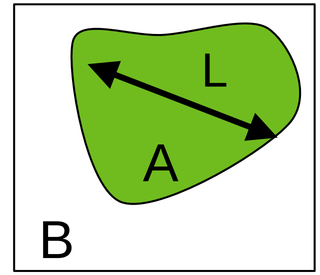
How to

DETECT if a given $|\psi\rangle$ is QSL STATE ?

compute its quantum entanglement

Entanglement entropy & area law

- Basic definitions
  - Reduced density matrix: $\rho_A = \mathrm{Tr}_B[|\psi\rangle\langle\psi|]$
  - Von Neumann entropy: $S_A = -\mathrm{Tr}_B[\rho_A \log \rho_A]$



$|\psi\rangle$

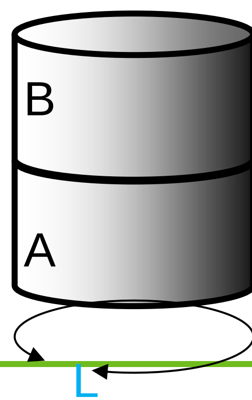
- Area law

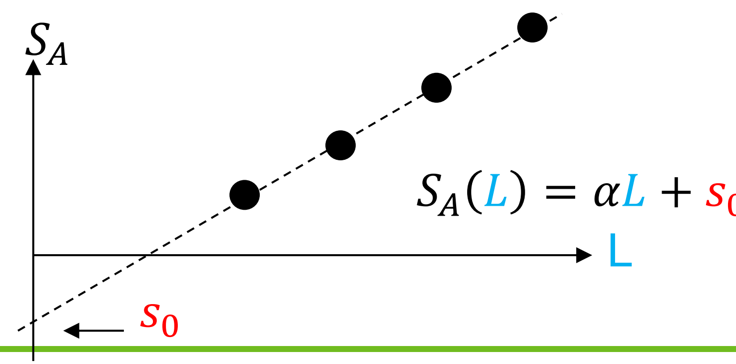
  The ground-state(s) (and low-energy excitations) of (many) Hamiltonians with short-ranged interactions have an entanglement entropy which scales like the area of the boundary of the subsystem

$$S_A \sim \mathcal{O}(\text{Area of } \partial A) = \mathcal{O}\left(L^{d-1}\right)$$

  Known gapless systems which violate the area law:
  - critical systems in $d = 1$
  - systems in $d > 1$ with a Fermi surface, where $S_A \sim \mathcal{O}\left(L^{d-1} \log L\right)$

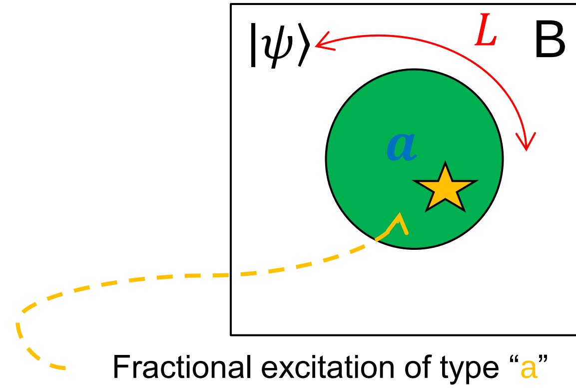
- Subleading corrections to the area law are often **universal**





Quantum Entanglement & topological order

The topological data of a QSL (anyon statistics, …) can be extracted from the ground states wave-functions.



Fractional excitation of type “a”

❑ Entanglement entropy ⇨ quantum dimensions

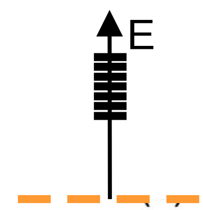
Levin & Wen, PRL 2006 & Kitaev & Preskill, PRL 2006

$$S_{a}(L) = \alpha L - \log\left(\frac{\mathcal{D}}{d_{a}}\right)$$

Total quantum dimension $\mathcal{D} = \sqrt{\sum_i (d_i)^2}$

Ground-state → trivial particle $\boldsymbol{a} = \mathbf{1} \rightarrow S_{g.-s.}(L) = \alpha L - \log(\mathcal{D})$

Remark: cannot make α arbitrary small (product state), since S must be ≥ 0.

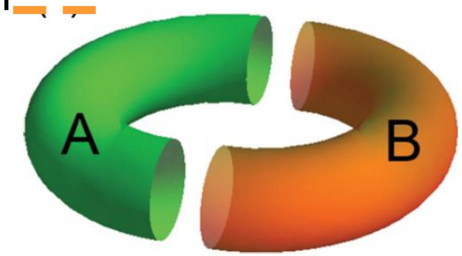


❑ Entanglement entropy ⇨ Braiding & statistics properties

“Quasiparticle statistics and braiding from ground-state entanglement”
Zhang, Grover, Turner, Oshikawa & Vishwanath PRB 2012

Idea: use Minimally Entangled States (MES)

MES are in one-to-one correspondence with the anyon types

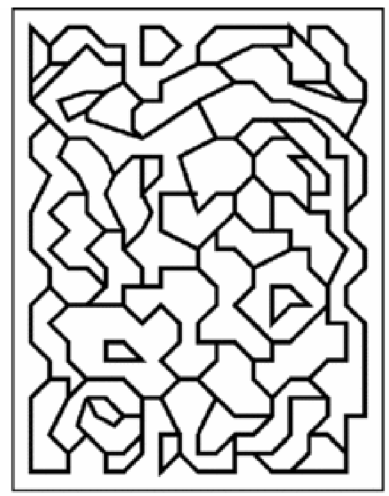


In the geometry above, the constant/subleading entropy term for the region A depends on the choice of a ground-state

Other gapped spin liquids – a few examples

- Chiral spin liquids → B. BAUER's talk
  - Close (bosonic) cousins of **fractional quantum Hall** phases
  - Original ideas: Kalmeyer & Laughlin, PRL 1987 and Wen PRB 1989
  - Gapped excitations in the bulk (abelian anyons) but ∃ gapless edge modes
  - Several realizations were recently discovered (thanks to 2D DMRG in particular) on **Kagome-lattice Heisenberg models** (with or without explicit T-reversal sym. breaking)
    - Y.-C. He, D. N. Sheng & Y. Chen, PRL 2014
    - Gong, Zhu & Sheng, Sci. Rep. 2014
    - Bauer, Cincio, Keller, Dolfi, Vidal, Trebst & Ludwig, Nat. Commun. 2014
    - W.-J. Hu, W. Zhu, Y. Zhang, S. Gong, F. Becca, D. N. Sheng, PRB 2014

- More types of anyons (possibly non-Abelian)
  - Toy models: string-nets Levin & Wen, PRB 2005
  - No realization in frustrated magnets ?



String-net condensed

Can a QSL be

GAPLESS ?

Parton construction (fermions)

- Abrikosov fermions

Review: P. Lee, N. Nagaosa & X.-G. Wen, RMP 2006

spin-½ operators: $\vec{S}_i = \frac{1}{2} c^\dagger_{i\mu} \vec{\sigma}_{\mu\nu} c_{i\nu} \quad \mu, \nu = \uparrow, \downarrow$

note: ∃ gauge redundancy

contraint: $\forall i \;\; c^\dagger_{i\uparrow} c_{i\uparrow} + c^\dagger_{i\downarrow} c_{i\downarrow} = 1$

- Mean-field decoupling (equiv. to $SU(2) \to SU(N)$ and large-N limit)

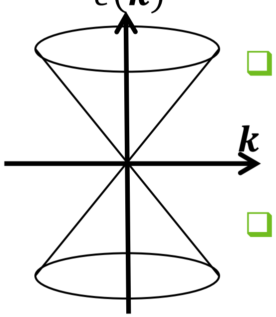
Heisenberg model $\to H_{\mathrm{MF}} = \sum_{<ij>} \chi_{ij} \left(c^\dagger_{i\uparrow} c_{j\uparrow} + c^\dagger_{i\downarrow} c_{j\downarrow} \right) + \eta_{ij} \left(c_{i\uparrow} c_{j\downarrow} - c_{i\downarrow} c_{j\uparrow} \right) + H.c.$

+ self consistency : $\chi_{ij} = \langle c^\dagger_{i\uparrow} c_{j\uparrow} + c^\dagger_{i\downarrow} c_{j\downarrow} \rangle \quad \eta_{ij} = \left\langle c^\dagger_{i\uparrow} c^\dagger_{j\downarrow} - c^\dagger_{i\downarrow} c^\dagger_{j\uparrow} \right\rangle \quad 1 = \langle c^\dagger_{i\uparrow} c_{i\uparrow} + c^\dagger_{i\downarrow} c_{i\downarrow} \rangle$

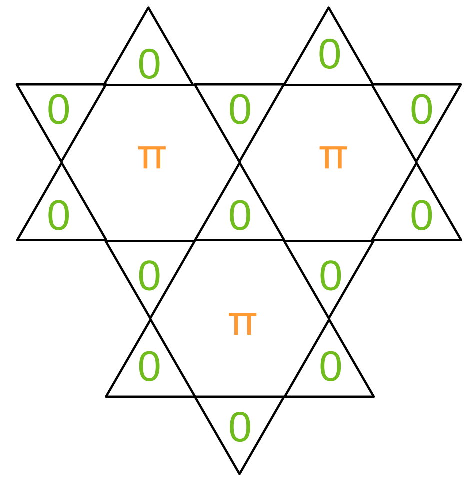
- Mean-field level
  - No magnetic long-range order at $T = 0$, rotationally invariant QSL
  - deconfined (free) spinon excitations
  - Potentially **gapless** (depends on the dispersion relation)
- Beyond mean-field
  - **Gutzwiller projection** (Monte-Carlo) to enforce the constrain exactly
    → variational states & optimization of the var. parameter (χ_{ij} & η_{ij})
  - Analyze the long-wavelength fluctuations:
    $\arg\left(\chi_{ij}\right)$ and $\arg\left(\eta_{ij}\right)$ are compact **gauge fields** (note: gauge group depends on the mean-field solution)
    Gauge fluctuations can drastically change (destroy) the mean-field picture:
    spinon confinement, translation symmetry breaking, …

Gapless QSL in 2d ?

- Dirac spinons coupled to U(1) gauge field – Algebraic Spin Liquids



  - Theory: stable phase for large-enough number of Dirac points
    M. Hermele, T. Senthil, M. P. A. Fisher, P. A. Lee, N. Nagaosa, and X.-G. Wen, PRB 2004
  - Maybe realized in the spin-½ Kagome Heisenberg model :
    Y. Ran, M. Hermele, P. A. Lee, and X. G. Wen, PRL 2007
    M. Hermele, Y. Ran, P. A. Lee, and X. G. Wen, PRB 2008
    Y. Iqbal, F. Becca, S. Sorella & D. Poilblanc, PRB 2013



- Fermi sea of spinons – Spin Bose metal

  Maybe realized in triangular organics ($\kappa-(ET)_2\,Cu_2(CN)_3$)
  & ring-exchange models on the triangular lattice:
  O. Motrunich, PRB 2005
  M. S. Block, D. N. Sheng, O. I. Motrunich & M. P. A. Fisher, PRL 2011

  (+many other refs on square ladders & triangular strips)

Gapless QSL in 3d

U(1) QSL phenomenology:

- Linearly dispersing gauge mode excitation (“photon”)
- Gapped deconfined “electric charges”, which interact via a $1/r$ potential (Coulomb)
- Gapped “magnetic charges” (monopoles), which interact via a $1/r$ potential (Coulomb)

- Examples:
  - S=1/2 Heisenberg antiferromagnet on the pyrochlore lattice
    in the limit of strong easy-axis exchange anisotropy.
    Hermele, Fisher & Balents PRB 2004
  - 3d QDM: Moesner & Sondhi PRB 2003
    See also: Raman, Moessner & Sondhi 2005

Summary

The theoretical understanding of QSL have made huge progress in the last ~10 years

- Growing list of models
- Important progress on the numerical front (DMRG, …)
- Relatively good understanding of the classification of 2d gapped SL in 2d (SET, …)
- Entanglement is a key probe for these phases of matter (TEE, MES, …)

- Experiments: many QSL candidates (Herbertsmithite, triangular organics, 3d Irridate, ..), but the connections with theories are still at some early stage.

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